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Algebraic Independence of Certain Power Series of Algebraic Numbers

KUMIKO NISHIOKA

Department of Mathematics, Nara Women's University
Kita-uoya Nishimachi, Nara 630, Japan

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Let $f(z)=\sum_{k=0}^{\infty} z^{k!}$. Then in p-adic field we prove that for any algebraic numbers $\alpha_1,\dots,\alpha_n$ with $0<|\alpha_i|<1$ $(1\leqslant i\leqslant n)$, $f(\alpha_1),\dots,f(\alpha_n)$ are algebraically independent over Q if and only if α_i/α_j is not a root of unity for $i\neq j$. In the complex field we prove the above result only when $n=2$, making use of the p-adic field. © 1986 Academic Press, Inc.

1. INTRODUCTION

The aim of this paper is to show the usefulness of the p-adic method in the study of the algebraic independence of certain power series of algebraic numbers. Let $f(z)=\sum_{k=0}^{\infty} a_k z^{e_k}$ be a power series with algebraic coefficients $a_k\neq 0$, the convergence radius $R>0$ and increasing integers e_k satisfying the condition

$$\lim_{k\to\infty} (e_k+\log M_k+\log A_k)\, S_k/e_{k+1}=0, \tag{1}$$

where $A_k=\max_{0\leqslant i\leqslant k} \overline{|a_i|}$, M_k is a positive integer such that $M_k a_i$ $(0\leqslant i\leqslant k)$ are algebraic integers and S_k is the degree of $Q(a_0,\dots,a_k)$ over Q. Then Cijsouw and Tijdeman [2] proved the transcendence of the number $f(\alpha)$ for any algebraic α with $0<|\alpha|<R$. Recently Bundschuh and Wylegala [1] showed that the numbers $f(\alpha_1),\dots,f(\alpha_n)$ are algebraically independent for any algebraic $\alpha_1,\dots,\alpha_n$ with $0<|\alpha_1|<\cdots<|\alpha_n|<R$.

In Section 2, we will prove that in the p-adic field, $f(\alpha_1),\dots,f(\alpha_n)$ are algebraically independent for any algebraic $\alpha_1,\dots,\alpha_n$ with $0<|\alpha_i|<R$, provided α_i/α_j is not a root of unity for $i\neq j$ and 0 is a limit point of $\{e_k\}_{k\geqslant 0}$ in Z_p. In Section 3, using the p-adic field we will show that in the complex field $\sum_{k=0}^{\infty}\alpha_1^{k!}$ and $\sum_{k=0}^{\infty}\alpha_2^{k!}$ are algebraically independent for any algebraic α_1, α_2 with $0<|\alpha_i|<1$ and α_1/α_2 not a root of 1. In Section 4, we

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will study power series $f(z)=\sum_{k=0}^{\infty} a_k z^k$ with algebraic coefficients $a_k \neq 0$, satisfying a suitable condition. We will prove that if algebraic $\alpha_1,\ldots,\alpha_n$ are distinct then $f(\alpha_1),\ldots,f(\alpha_n)$ are algebraically independent whether the field is Archimedean or not.

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Let p be ∞ or a prime number. By C_p we denote the complex number field or the completion of the algebraic closure of the p-adic number field Q_p according to whether p is ∞ or a prime number. We denote the absolute value of C_p by $|\cdot|_p$. For a power series $f(z)$ with algebraic coefficients and an algebraic number α, $f(\alpha)_p$ denotes the value of $f(z)$ at α in C_p if it converges. For an algebraic number α, we denote by $\overline{|\alpha|}$ the maximum of the absolute values of the conjugates of α and by $\operatorname{den}(\alpha)$ the smallest positive rational integer d such that $d\alpha$ is an algebraic integer. Let $f(z)=\sum_{k=0}^{\infty} a_k z^{e_k}$ be a power series with algebraic coefficients $a_k \neq 0$ and the convergence radius R_p in C_p. We put $A_k = \max_{0 \leqslant i \leqslant k} \overline{|a_i|}$, $S_k = [Q(a_0,\ldots,a_k)\colon Q]$, the degree of $Q(a_0,\ldots,a_k)$ over Q. Further M_k will denote the least positive integer such that $M_k a_i (0 \leqslant i \leqslant k)$ are algebraic integers.

THEOREM 1. *Let p be a prime number. Assume that*

$$\lim_{k\to\infty} \frac{(e_k + \log M_k + \log A_k)\, S_k}{\log |a_{k+1}|_p + e_{k+1} \log r} = 0, \tag{2}$$

for any positive $r < R_p$ and 0 is a limit point of $\{e_k\}_{k \geqslant 0}$. Let $\alpha_1,\ldots,\alpha_n$ be algebraic numbers with $0<|\alpha_i|_p<R_p$, and α_i/α_j be not a root of unity for $i \neq j$. Then $f^{(l)}(\alpha_i)_p$ $(1 \leqslant i \leqslant n,\ 0 \leqslant l)$ are algebraically independent over Q, where $f^{(l)}(z)$ denotes the lth *derivative of $f(z)$.*

Remark. If $f(z)$ satisfies the condition (1) in Section 1, then $f(z)$ satisfies the condition (2) in Theorem 1.

COROLLARY. *Suppose that p is a prime number, the a_k are rational integers, $\lim_{k\to\infty} e_k/e_{k+1}=0$, and 0 is a limit point of $\{e_k\}_{k\geqslant 0}$ in Z_p. Let $\alpha_1,\ldots,\alpha_n$ be algebraic numbers with $0<|\alpha_i|_p<R_p$, and α_i/α_j be not a root of unity for $i\neq j$. Then $f^{(l)}(\alpha_i)_p$ $(1\leqslant i\leqslant n,\ 0\leqslant l)$ are algebraically independent over Q.*

EXAMPLE. Let p be a prime number, $f(z)=\sum_{k=0}^{\infty} z^{k!}$, and $\alpha_1,\ldots,\alpha_n$ be algebraic numbers with $0<|\alpha_i|_p<1$. Then $f^{(l)}(\alpha_i)_p$ $(1\leqslant i\leqslant n,\ 0\leqslant l)$ are algebraically independent if and only if α_i/α_j is not a root of unity for $i\neq j$.

THEOREM 2. *Assume that the condition* (2) *in the Theorem* 1 *is satisfied and* $\alpha_1,\ldots,\alpha_n$ *be algebraic numbers with* $0<|\alpha_i|_p<R_p$. *Further, suppose* $f^{(l)}(\alpha_i)_q$ $(1\leqslant i\leqslant n,\ 0\leqslant l)$ *converge and are algebraically independent over* Q *in some field* C_q. *Then* $f^{(l)}(\alpha_i)_p$ $(1\leqslant i\leqslant n,\ 0\leqslant l)$ *are algebraically independent over* Q.

EXAMPLE. Let $f(z)=\sum_{k=0}^{\infty}(2/3)^{(k+1)!}z^{k!}$, and $\alpha_1,\ldots,\alpha_n$ be algebraic numbers with $\alpha_i\neq 0$. Taking $p=\infty$, $q=2$ in Theorem 2, we obtain that $f^{(l)}(\alpha_i)_\infty$ $(1\leqslant i\leqslant n,\ 0\leqslant l)$ are algebraically independent if and only if α_i/α_j is not a root of unity for $i\neq j$.

In what follows, c_1, $c_2,\ldots,$ denote positive constants depending only on $f(z)$ and $\alpha_1,\ldots,\alpha_n$.

Proof of Theorem 1. Let $\lambda>\frac{1}{2}$ be arbitrary, take and fix r such that $|\alpha_i|_p<r<R_p$ for any i. Attending to $\lim_{k\to\infty}(\log|a_{k+1}|_p+e_{k+1}\log r)=-\infty$,

$$\log|a_{k+1}|_p+e_{k+1}\log r\leqslant-\lambda(e_k+\log M_k+\log A_k)\,S_k,$$

if k is sufficiently large. Therefore for sufficiently large k

$$\begin{aligned}&e_{k+1}(e_{k+1}-1)\cdots(e_{k+1}-l+1)|a_{k+1}|_p|\alpha_i|_p^{e_{k+1}}\\&\quad\leqslant e^{-\lambda(e_k+\log M_k+\log A_k)S_k},\end{aligned}$$

and so for sufficiently large m,

$$\begin{aligned}&\left|\sum_{k=m+1}^{\infty}e_k(e_k-1)\cdots(e_k-l+1)\,a_k\alpha_i^{e_k}\right|_p\\&\quad\leqslant\sum_{k=m+1}^{\infty}e^{-\lambda\{(e_m+\log M_m+\log A_m)S_m+k-m-1\}}\\&\quad\leqslant c_1e^{-\lambda(e_m+\log M_m+\log A_m)S_m}.\end{aligned}\tag{3}$$

We prove the theorem by induction on n. If $n=0$, then the theorem is true. We suppose $n>0$ and $f^{(l)}(\alpha_i)_p$ $(1\leqslant i\leqslant n,\ 0\leqslant l\leqslant L)$ are algebraically dependent over Q. Put

$$\Xi=(f(\alpha_1)_p,\ \alpha_1f^{(1)}(\alpha_1)_p,\ldots,\ \alpha_n^Lf^{(L)}(\alpha_n)_p).$$

Then there is a polynomial $F\in Z[Y_{10},\ Y_{11},\ldots,\ Y_{nL}]$ such that $F(\Xi)=0$. We may assume F has the least total degree among them. By the assumption of induction, for any i, there exists a number l such that $\partial F/\partial Y_{il}\neq 0$, and so $\partial F/\partial Y_{il}(\Xi)\neq 0$. Put

$$u(i, l, m) = \sum_{k=0}^{m} e_k(e_k - 1) \cdots (e_k - l + 1)\, a_k \alpha_i^{e_k}$$

and

$$U_m = (u(1, 0, m), u(1, 1, m), ..., u(n, L, m)).$$

Then $\lim_{m \to \infty} U_m = \Xi$ and

$$F(U_m) = F(U_m) - F(\Xi) = \sum_{|J| \geqslant 1} \frac{1}{J!} \frac{\partial^{|J|} F}{\partial y^J} (\Xi)(U_m - \Xi)^J,$$

where

$$J = (j_{10}, j_{11}, ..., j_{nL}),$$

$$|J| = \sum_{i=1}^{n} \sum_{l=0}^{L} j_{il},$$

$$J! = \prod_{i=1}^{n} \prod_{l=0}^{L} j_{il}!,$$

$$\frac{\partial^{|J|}}{\partial y^J} = \prod_{i=1}^{n} \prod_{l=0}^{L} \frac{\partial^{j_{il}}}{\partial y_{il}^{j_{il}}},$$

and

$$(U_m - \Xi)^J = \prod_{i=1}^{n} \sum_{l=0}^{L} \{u(i, l, m) - \alpha_i^l f^{(l)}(\alpha_i)_p\}^{j_{il}}.$$

Then by (3) we have

$$|F(U_m)|_p \leqslant c_2 e^{-\lambda(e_m + \log M_m + \log A_m) S_m}, \tag{4}$$

if m is sufficiently large. On the other hand,

$$\begin{aligned} \overline{|F(U_m)|} &\leqslant c_3 \{(m+1)\, e_m^L A_m c_4^{e_m}\}^{c_5}, \\ d(F(U_m)) &\leqslant (M_m c_6^{e_m})^{c_5}, \\ [Q(F(U_m)) : Q] &\leqslant c_7 S_m. \end{aligned} \tag{5}$$

Suppose that there are infinitely many m with $F(U_m) \neq 0$. To $F(U_m)$ we apply the fundamental inequality: For any algebraic $\alpha \neq 0$, $\log |\alpha|_p \geqslant -[Q(\alpha) : Q]\{\log \overline{|\alpha|} + \log d(\alpha)\}$. If m is sufficiently large and $F(U_m) \neq 0$, then by (4) and (5), we have

$$-\lambda(e_m + \log M_m + \log A_m)\, S_m \geqslant -c_8(e_m + \log M_m + \log A_m)\, S_m,$$

and so $\lambda \leqslant c_8$. Since λ is any positive number greater than $\frac{1}{2}$, we have a contradiction. Hence there exists M such that $F(U_m)=0$ for any $m \geqslant M$. We may assume

$$|\alpha_1|_p = \cdots = |\alpha_t|_p > |\alpha_{t+1}|_p \geqslant \cdots \geqslant |\alpha_n|_p.$$

We put $\beta_i = \alpha_i/\alpha_1$, $i=1,..., t$. Since $|\beta_i|_p = 1$, we can put $\beta_i = \zeta_i \gamma_i$, where $|\gamma_i - 1|_p < 1$ and ζ_i is a root of unity. Since γ_i/γ_j is not a root of unity for $i \neq j$, $\log_p \gamma_1,..., \log_p \gamma_t$ are distinct to each other, where $\log_p(\cdot)$ denotes the p-adic logaritmic function. Fix a positive integer N such that $\zeta_i^N = 1$ for any i. By the assumption, we can take a subsequence $\{e_{m(h)}\}_{h \geqslant 0}$ of $\{e_k\}_{k \geqslant 0}$ and a rational integer w such that $\lim_{h \to \infty} e_{m(h)} = 0$ in C_p and $e_{m(h)} \equiv w \bmod N$ for any h. If $m(h) > M$, then

$$\begin{aligned}
0 &= F(U_{m(h)-1}) - F(U_{m(h)}) \\
&= \sum_{|J| \geqslant 1} \frac{(-1)^{|J|}}{J!} \frac{\partial^{|J|} F}{\partial Y^J} (U_{m(h)}) \\
&\quad \times \prod_{i=1}^{n} \prod_{l=0}^{L} \{e_{m(h)}(e_{m(h)}-1) \cdots (e_{m(h)} - l + 1)\, a_{m(h)} \alpha_i^{e_{m(h)}}\}^{j_{il}} \\
&= \sum_{|J| \geqslant 1} B(h, J) \qquad \text{(say)}.
\end{aligned}$$

Put

$$\begin{aligned}
S_1(h) &= \sum_{|J| \geqslant 2} B(h, J), \\
S_2(h) &= \sum_{|J| = 1 \text{ and } j_{il} = 1 \text{ for } i > t} B(h, J), \\
S_3(h) &= \sum_{|J| = 1 \text{ and } J_{il} = 1 \text{ for } i \leqslant t} B(h, J).
\end{aligned}$$

Then by the fact $\lim_{h \to \infty} (\partial^{|J|} F/\partial Y^J)(U_{m(h)}) = (\partial^{|J|} F/\partial Y^J)(\Xi)$, we have

$$\begin{aligned}
|S_1(h)|_p &\leqslant c_9 |a_{m(h)} \alpha_1^{e_{m(h)}}|_p^2, \\
|S_2(h)|_p &\leqslant c_{10} |a_{m(h)} \alpha_{t+1}^{e_{m(h)}}|_p.
\end{aligned} \tag{6}$$

Define

$$\begin{aligned}
T(X) &= -\sum_{i=1}^{t} \sum_{l=0}^{L} \frac{\partial F}{\partial Y_{il}} (\Xi)\, X \cdots (X - l + 1)\, \zeta_i^w e^{X \log_p \gamma_i} \\
&= \sum_{k=0}^{\infty} Q_k X^k,
\end{aligned}$$

where $Q_k \in C_p$. Since all of $(\partial F/\partial Y_{il})$ (Ξ) $(1 \leqslant i \leqslant t,\ 0 \leqslant l \leqslant L)$ are not zero, there is a nonnegative integer v such that $Q_v \neq 0$. Let v be the least among such integers. Then

$$|T(e_{m(h)})|_p \geqslant c_{11} |e_{m(h)}|_p^v, \tag{7}$$

for sufficiently large h. On the other hand

$$\begin{aligned} &S_3(h)/a_m(h)\, \alpha_1^{e_{m(h)}} \\ &\quad = -\sum_{i=1}^{t} \sum_{l=0}^{L} \frac{\partial F}{\partial Y_{il}} (U_{m(h)})\, e_{m(h)} \cdots (e_{m(h)} - l + 1)\, \zeta_i^w e^{e_{m(h)} \log_p \gamma_i}, \end{aligned}$$

and so

$$\begin{aligned} &|T(e_{m(h)}) - S_3(h)/a_{m(h)} \alpha_1^{e_{m(h)}}|_p \\ &\quad \leqslant c_{13} e^{-\lambda(e_{m(h)} + \log M_{m(h)} + \log A_{m(h)}) S_{m(h)}}, \end{aligned} \tag{8}$$

if h is sufficiently large. By (7) and (8), we have

$$|S_3(h)/a_{m(h)} \alpha_1^{e_{m(h)}}|_p \geqslant c_{14} e_{m(h)}^{-v}, \tag{9}$$

since $|e_{m(h)}|_p^v \geqslant e_{m(h)}^{-v}$. By the fact $S_1(h) + S_2(h) + S_3(h) = 0$, (6) and (9), we have

$$c_{14} |a_{m(h)} \alpha_1^{e_{m(h)}}|_p\, e_{m(h)}^{-v} \leqslant c_9 |a_{m(h)} \alpha_1^{e_{m(h)}}|_p^2 + c_{10} |a_{m(h)} \alpha_{t+1}^{e_{m(h)}}|_p.$$

The facts

$$\lim_{h \to \infty} |(\alpha_{t+1}/\alpha_1)^{e_{m(h)}}|_p\, e_{m(h)}^v = 0$$

and

$$\lim_{h \to \infty} |a_{m(h)} \alpha_1^{e_{m(h)}}|_p\, e_{m(h)}^v = 0$$

imply $c_{14} \leqslant 0$. This contradicts that c_{14} is a positive constant. Thus we have proved the theorem.

Proof of Theorem 2. Define Ξ, F, and U_m as the proof of Theorem 1. Then we see $F(U_m) = 0$ for any $m \geqslant M$ by the same way as the proof of Theorem 1. As $m \to \infty$ in C_q, we get

$$F(f(\alpha_1)_q, \alpha_1 f^{(1)}(\alpha_1)_q, \ldots, \alpha_n^L f^{(L)}(\alpha_n)_q) = 0.$$

This is a contradiction to our assumption, and the theorem is proved.

3

Let $f(z)=\sum_{k=0}^{\infty} z^{k!}$. In general, we can not deduce the algebraic independence of $f(\alpha_1)_\infty,\dots, f(\alpha_n)_\infty$ for algebraic numbers $\alpha_1,\dots,\alpha_n$ from Theorem 2. But if $n=2$, we can prove:

THEOREM 3. *Let α_1, α_2 be algebraic numbers with $0<|\alpha_i|_\infty<1$, $i=1,2$. Then $f(\alpha_1)_\infty$ and $f(\alpha_2)_\infty$ are algebraically independent over Q if and only if α_1/α_2 is not a root of unity.*

Proof of Theorem 3. Put $f(\alpha_i)_\infty=\xi_i$, $i=1,2$. Suppose that ξ_1 and ξ_2 are algebraically dependent over Q. Then there exists a polynomial $F\in\bar{Q}[x_1,x_2]$ such that $F(\xi_1,\xi_2)=0$. We may assume F has the least total degree among them. Therefore $F(x_1,x_2)$ is absolutely irreducible and $(\partial F/\partial X_i)\,(\xi_1,\xi_2)\neq 0$, $i=1,2$. Set

$$u(i,m)=\sum_{k=0}^{m}\alpha_i^{k!}.$$

By the same way as the proof of Theorem 1, we have

$$F(u(1,m),u(2,m))=0$$

for any $m\geqslant M$. Fix a prime number p such that $|\alpha_1|_p=|\alpha_2|_p=1$. Multiplying a power root of 1 to α_i if necessary, we may assume $|\alpha_i-1|_p<1$, $i=1,2$. Since $|u(i,m)|_p\leqslant 1$ for any m, we can take a subsequence $\{m(h)\}_{h\geqslant 0}$ of $\{m\}_{m\geqslant 0}$ such that $\{u(i,m(h))\}_{h\geqslant 0}$ converges to ξ_i' (say), $i=1,2$, in C_p. If $m(h)\geqslant M$, then for any positive integer n,

$$F(u(1,m(h)+n),\ u(2,m(h)+n))=0,$$

where

$$u(i,m(h)+n)=u(i,m(h))+\alpha_i^{(m(h)+1)!}+\cdots+\alpha_i^{(m(h)+n)!}$$

Tending h to ∞, we obtain

$$F(\xi_1'+n,\xi_2'+n)=0.$$

Hence the algebraic curve V: $F(x_1,x_2)=0$ contains the line V': $x_1-x_2=\xi_1'-\xi_2'$. Since F is absolutely irreducible, $V=V'$ and so

$$F(x_1,x_2)=ax_1+bx_2+c \qquad \text{(say)}.$$

Then for any $m\geqslant M$

$$\begin{aligned}0&=F(u(1,m),u(2,m))-F(u(1,m+1),u(2,m+1))\\&=a\alpha_1^{m!}+b\alpha_2^{m!}.\end{aligned}$$

This contradicts that α_1/α_2 is not a root of unity, and therefore the theorem is proved.[1]

4

Let $f(z)=\sum_{k=0}^{\infty} a_k z^k$ be a power series with coefficients $a_k \neq 0$ and convergence radius R_p in C_p. We define S_k, A_k and M_k as same as Section 2.

THEOREM 4. *Assume*

$$\lim_{k\to\infty} \frac{(k+\log M_k+\log A_k)\,S_k}{\log|a_{k+1}|_p}=0. \tag{10}$$

Let $\alpha_1,\dots,\alpha_n$ be algebraic numbers with $0<|\alpha_i|_p<R_p$ and distinct to each other. Then $f^{(l)}(\alpha_i)_p$ $(1\leqslant i\leqslant n,\ 0\leqslant l)$ are algebraically independent over Q.

EXAMPLE. Let a be an algebraic number with $0<|a|_p<1$ and $f(z)=\sum_{k=0}^{\infty} a^{k!}z^k$. Then for any nonzero algebraic numbers which are distinct to each other, $f^{(l)}(\alpha_i)_p$ $(1\leqslant i\leqslant n,\ 0\leqslant l)$ are algebraically independent over Q.

Proof of Theorem 4. Define Ξ, F, and U_m as in the proof of Theorem 1. By the assumption (10), we know the condition (2) in Section 2 is satisfied with $e_k=k$ and so $F(U_m)=0$ for any $m\geqslant M$. We may assume

$$|\alpha_1|_p=|\alpha_2|_p=\cdots=|\alpha_t|_p>|\alpha_{t+1}|_p\geqslant\cdots\geqslant|\alpha_n|_p.$$

If $m>M$, then

$$\begin{aligned}0&=F(U_{m-1})-F(U_m)\\&=\sum_{|J|\geqslant 1}\frac{(-1)^{|J|}}{J!}\frac{\partial^{|J|}F}{\partial y^J}(U_m)\\&\quad\times\sum_{i=1}^{n}\sum_{l=0}^{L}\{m(m-1)\cdots(m-l+1)\,a_m\alpha_i^m\}^{j_{il}}\\&=\sum_{|J|\geqslant 1} B(m,J)\qquad(\text{say}).\end{aligned}$$

[1] D. Masser pointed out that he had another proof of Theorem 3 not depending on p-adic method and W. Adams (*J. Pure Appl. Algebra* **13** (1978), 41–47) had introduced at first p-adic considerations.

Put

$$S_1(m) = \sum_{|J| \geqslant 2} B(m, J),$$

$$S_2(m) = \sum_{|J| = 1 \text{ and } J_{il} = 1 \text{ for } i > t} B(m, J),$$

$$S_3(m) = \sum_{|J| = 1 \text{ and } j_{il} = 1 \text{ for } i \leqslant t} B(m, J).$$

Then we have

$$\begin{aligned} |S_1(m)|_p &\leqslant c_{15} m^{2L} |a_m \alpha_1^m|_p^2, \\ |S_2(m)|_p &\leqslant c_{16} m^L |a_m \alpha_{t+1}^m|_p. \end{aligned} \tag{11}$$

First, we consider the case $p = \infty$. We use Turán's theorem.

THEOREM (Turán [4, p. 53]). *If $b_1, ..., b_t$ are any complex numbers and $\beta_1, ..., \beta_t$ are nonzero complex numbers which are distinct to each other, then for any nonnegative integer m,*

$$\max_{1 \leqslant y \leqslant t} \frac{|b_1 \beta_1^{m+y} + \cdots + b_t \beta_t^{m+y}|}{|b_1| |\beta_1|^{m+y} + \cdots + |b_t| |\beta_t|^{m+y}} \geqslant c_{17},$$

where c_{17} is a positive constant depending only on $\beta_1, ..., \beta_t$.

Put $\beta_i = \alpha_i / \alpha_1$ for $i = 1, ..., t$. Then

$$S_3(m)/a_m \alpha_1^m = -\sum_{i=1}^{t} \sum_{l=0}^{L} \frac{\partial F}{\partial y_{il}} (U_m)\, m(m-1) \cdots (m-l+1)\, \beta_i^m.$$

Define

$$L' = \max \left\{ l \,\middle|\, \frac{\partial F}{\partial y_{il}} (\Xi) \neq 0 \text{ for some } i \leqslant t. \right\},$$

$$T_1(m) = -\sum_{i=1}^{t} \frac{\partial F}{\partial y_{iL'}} (U_m)\, m(m-1) \cdots (m-L'+1)\, \beta_i^m,$$

$$T_2(m) = -\sum_{i=1}^{t} \sum_{l=0}^{L'-1} \frac{\partial F}{\partial y_{il}} (U_m)\, m(m-1) \cdots (m-l+1)\, \beta_i^m,$$

$$T_3(m) = -\sum_{i=1}^{t} \sum_{l=L'+1}^{L} \frac{\partial F}{\partial y_{il}} (U_m)\, m(m-1) \cdots (m-l+1)\, \beta_i^m,$$

and

$$T(m, y) = -\sum_{i=1}^{t} \frac{\partial F}{\partial y_{iL'}}(\Xi)\, m(m-1)\cdots(m-L'+1)\,\beta_i^{m+y}.$$

Then we have

$$S_3(m)/a_m\alpha_1^m = T_1(m) + T_2(m) + T_3(m),$$

$$|T_1(m+y) - T(m, y)|_\infty$$
$$\begin{cases} \leqslant c_{18} e^{-\lambda(m+\log M_m + \log A_m)S_m} & \text{if } L'=0, \\ \leqslant c_{18} e^{-\lambda(m+\log M_m+\log A_m)S_m} + c_{19} m^{L'-1} & \text{if } L'>0, \end{cases}$$

$$|T_2(m+y)|_\infty \begin{cases} =0 & \text{if } L'=0, \\ \leqslant c_{20} m^{L'-1} & \text{if } L'>0, \end{cases}$$

and

$$|T_3(m+y)|_\infty \leqslant c_{21} e^{-\lambda(m+\log M_m+\log A_m)S_m} m^L.$$

By Turán's theorem,

$$\max_{1 \leqslant y \leqslant t} |T(m, y)|_\infty \geqslant c_{22} m^{L'},$$

and so

$$\max_{1 \leqslant y \leqslant t} |S_3(m+y)/a_{m+y}\alpha_1^{m+y}|_\infty \geqslant c_{23} m^{L'}.$$

Therefore for infinitely many m,

$$|S_3(m)|_\infty \geqslant c_{24} m^{L'} |a_m \alpha_1^m|_\infty. \tag{12}$$

By the fact $S_1(m) + S_2(m) + S_3(m) = 0$, (11) and (12), we have

$$c_{24} m^{L'} |a_m\alpha_1^m|_\infty \leqslant c_{15} m^{2L} |a_m\alpha_1^m|_\infty^2 + c_{16} m^L |a_m \alpha_{t+1}^m|_\infty,$$

for infinitely many m. Hence $c_{24} \leqslant 0$ and this is a contradiction.

Second, we consider the case $p \neq \infty$. Let $\beta_i = \alpha_i/\alpha_1$ for $i = 1, \ldots, t$. Then β_i a p-adic unit and $\beta_i = \zeta_i \gamma_i$, where ζ_i is a root of unity and $|\gamma_i - 1|_p < 1$. Take a positive number N such that $\zeta_i^N = 1$ for any i and $|N|_p$ is sufficiently small. Put

$$T(g, h) = -\sum_{i=1}^{t} \sum_{l=0}^{L} \frac{\partial F}{\partial y_{il}}(\Xi)$$
$$\times (g+hN)\cdots(g+hN-l+1)\,\zeta_i^g \gamma_i^g e^{hN \log_p \gamma_i},$$

where $0 \leqslant g \leqslant N-1$ and $h \geqslant 0$. Then at least one of $T(g, h)$ $(0 \leqslant g \leqslant N-1)$ is not identically zero with respect to h, since the $\zeta_i \gamma_i$ are distinct to each other and one of the $(\partial F/\partial y_{il})(\Xi)$ is not zero. Let $T(g, h)$ be one of these. Then for some nonnegative integer v

$$|T(g, h)|_p \geqslant c_{25} h^{-v}$$

if $|h|_p$ is sufficiently small. On the other hand

$$S_3(m)/a_m \alpha_1^m = -\sum_{i=1}^{I} \sum_{l=0}^{L} \frac{\partial F}{\partial y_{il}} (U_m) \times m \cdots (m-l+1)\, \zeta_i^g \gamma_i^g e^{hN \log_p \gamma_i},$$

and

$$|S_3(m)/a_m \alpha_1^m - T(g, h)|_p \leqslant c_{26} e^{-\lambda(m + \log M_m + \log A_m) S_m},$$

where $m = g + hN$ and $|h|_p$ is sufficiently small. Therefore

$$|S_3(g+hN)|_p \geqslant c_{26} |a_{g+hN} \alpha_1^{g+hN}|_p (g+hN)^{-v} \tag{13}$$

if $|h|_p$ is sufficiently small. By (11) and (13), we have $c_{26} \leqslant 0$ and this is a contradiction. Hence the theorem is proved.

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